HALL'S COHERENT STATES, THE CAMERON–MARTIN THEOREM, AND THE QUANTIZATION OF YANG–MILLS THEORY ON A CIRCLE[1]

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Abstract

We discuss the classical and quantum reduction to the space of physical degrees of freedom of Yang–Mills theory on a circle (so that space-time is a cylinder). Although the classical reduced phase space is finite-dimensional, the quantum reduction procedure is mathematically fascinating, involving firstly the Wiener measure on a loop group, secondly a generalization of the Cameron–Martin theorem to loop groups, and thirdly Hall's coherent states for compact Lie groups. Our approach is based on a quantum analogue of the classical Marsden–Weinstein symplectic reduction process.

1 . INTRODUCTION

Yang–Mills theories on a two-dimensional space-time serve as a laboratory for studying issues that are relevant to general gauge theories. Even without fermions, one may look at the role of gauge invariance, constraints, reduction, observables, the geometry of the space of physical degrees of freedom, possible singularities occurring in the latter, and what not. Here it is of

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particular interest to understand how the structure of the classical theory is reflected in the quantum theory. If one includes fermions, one may in addition look at anomalies, spectral flow, supersymmetry, etc. Moreover, there are unexpected connections to string theory and M-theory [5, 6].

The advantage of the two-dimensionality of the model is that, with the present state of the art, one may proceed in a mathematically rigorous fashion. The fact that the model is physically somewhat trivial then turns out to be compensated by an astonishingly rich mathematical structure. On the geometry side, this seems to have been first realized by Witten [31, 32], who considered the Euclidean version in which the the theory is defined on a Riemann surface (also cf. [28, 29]). As we shall see, even the much simpler Minkowski formulation on a circle, in which space-time is taken to be a cylinder, involves matters of a certain interest to analysis and measure theory [20, 34, 35, 19].

In what follows, we shall always speak about this particular version of Yang–Mills theory. Without loss we will work in the temporal gauge $A_0 = 0$, in which the residual gauge transformations are time-independent. Thus the gauge group consists of loops in the structure group K. Since in this paper we will not discuss the singular structure of the physical theory (cf. [33, 35]), we further specialize to the group of based gauge transformations, which consists of loops in K that start and end at the unit element e.

Our central concern in this paper is the reduction from the degrees of freedom that are originally given (viz. the space of all gauge fields and their conjugate momenta) to the space of physical degrees of freedom. As first remarked by Rajeev [23], the reduced phase space of the classical theory is finite-dimensional, being canonically isomorphic to the cotangent bundle T^*K. As in four-dimensional Yang–Mills theory [3], one may see this reduction as a special instance of a Marsden–Weinstein quotient in symplectic geometry [1, 19]. While this neat geometric fact does not dramatically change the perspective on the classical theory, it lies at the basis of our quantization procedure. For we are going to quantize the theory by first quantizing the unconstrained phase space, and subsequently implementing a recent proposal [17, 18] to quantize the Marsden–Weinstein reduction process by a technique based on the theory of induced representations of C^*-algebras [19] (for different rigorous approaches see [7, 4, 14]).

It turns out that this technique can be carried through in the case at hand, involving an integration over the gauge group. Now, two interesting things happen. Firstly, the physical theory turns out to be gauge invariant as a consequence of the Cameron-Martin theorem for loop groups [9, 22]. We will provide an interesting perspective on this theorem and its application to quantum gauge theories through the formalism of Hilbert subspaces of locally convex vector spaces [26, 27, 30]. Secondly, the 'quantum reduction map' from the unphysical state space to the physical Hilbert space turns out

to map coherent states for the gauge field A into Hall's coherent states [11], parametrized by the Wilson loop $\mathcal{W}(A) \in K$.

Let us now turn to the details. In section 2 we explain the application of Marsden–Weinstein reduction to Yang–Mills theory. In section 3 we take the first steps towards quantizing this procedure. Section 4 contains an intermezzo on Hilbert subspaces and measures on infinite-dimensional spaces. In section 5 we adapt and apply this material to Yang–Mills theory. In section 6 we review Hall's coherent states, placing them in a general context. Finally, section 7 outlines how the various threads come together.

2. MARSDEN-WEINSTEIN REDUCTION IN GAUGE THEORY

The starting point is a (strongly) symplectic manifold S, mathematically representing the phase space of the system, with associated Poisson bracket. A Lie group $\mathcal{G}$ is supposed to act on S in strongly Hamiltonian fashion. This means two things: firstly, the action is canonical, in that the Poisson brackets are preserved, and secondly, the action is infinitesimally generated in the following sense. For each X in the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ there is a function $J_X \in C^\infty(S)$ such that $\{J_X, f\} = \xi_X f$, where $\xi_X f(\sigma) = df(\mathrm{Exp}(tX)\sigma)/dt(t=0)$. The map $X \mapsto J_X$ is automatically linear, defining a map $J : S \to \mathfrak{g}^*$ by $\langle J(\sigma), X\rangle = J_X(\sigma)$. It is then required that this map be equivariant with respect to the $\mathcal{G}$-action on S and the coadjoint action on $\mathfrak{g}^*$. Equivalently, J is a Poisson map relative to the Lie–Poisson structure on $\mathfrak{g}^*$. One calls J a momentum map for the given $\mathcal{G}$-action on S. The Marsden–Weinstein quotient

$$S^0 = J^{-1}(0)/\mathcal{G} \tag{2.1}$$

is then a symplectic manifold, provided that the $\mathcal{G}$-action on S is proper and free. See [1, 19] for details.

This formalism has two applications to physics, which are identical in the underlying mathematics, but have a different physical interpretation. In the oldest one, $\mathcal{G}$ is the symmetry group of some Hamiltonian h on S. Noether's theorem then states that each J_X is conserved under the flow generated by h, which quotients to a well-defined function h^0 on the reduced phase space S^0. One then attempts to solve the equations of motion on S by finding the flow of h^0 on S^0, and subsequently constructing a preimage of this flow on S. In a more modern application, $\mathcal{G}$ is a gauge group, and S is the phase space of unconstrained degrees of freedom of a gauge theory. The reduced space S^0 is then the true phase space of physical degrees of freedom of the system. This application of symplectic reduction was first considered by J. Marsden and his school; see [3]. Typically, the constraint manifold $J^{-1}(0)$ consists of those fields that satisfy Gauss's law.

To apply this to Yang–Mills theory on the circle $\mathbb{T}$, we take the configuration space of connections on the trivial principal K-bundle $P = \mathbb{T} \times K$

to be the real Hilbert space $\mathcal{A} = L^2(\mathbb{T}, \mathfrak{k})$, where $\mathfrak{k}$ is the Lie algebra of the compact structure group K. Identifying $\mathfrak{k}$ with its dual $\mathfrak{k}^*$ through the choice of an invariant inner product, the phase space $T^*\mathcal{A}$ may be identified with the complex Hilbert space

$$S = L^2(\mathbb{T}, \mathfrak{k}_{\mathbb{C}}) \tag{2.2}$$

of complexified L^2-connections. The gauge group $\mathcal{G}$ of this theory is the real Hilbert manifold $\mathcal{H}_1(\mathbb{T}, K)$, consisting of all based loops $g \in C(\mathbb{T}, K)$ whose (weak) derivative $\dot{g} = g^{-1}dg/dt$ lies in $L^2(\mathbb{T}, \mathfrak{k})$. We write $Z = A + \frac{1}{2}iE$.

The action

$$g : Z \mapsto Z^g = \mathrm{Ad}(g)Z + gdg^{-1} \tag{2.3}$$

of $\mathcal{G} \ni g$ on S, which is the pullback of the usual $\mathcal{G}$-action on $\mathcal{A}$, may be shown to be smooth, proper, and free [24], as well as strongly Hamiltonian, with momentum map J [20, 19]. Hence the Marsden–Weinstein quotient S^0 is a symplectic manifold, which may be explicitly computed to be canonically isomorphic to the cotangent bundle T^*K. To understand why, it is convenient to introduce the complexification $K_{\mathbb{C}}$ of K. This is a Lie group whose Lie algebra is $\mathfrak{k}_{\mathbb{C}}$ (the complexification of $\mathfrak{k}$), and which is diffeomorphic to T^*K [12]. The pertinent diffeomorphism equips $K_{\mathbb{C}}$ with a symplectic structure, borrowed from the canonical one on T^*K. Now recall the Wilson loop

$$\mathcal{W}(A) = P\,\mathrm{Exp}\left(-\oint_{\mathbb{T}} A\right), \tag{2.4}$$

seen as a map $\mathcal{W} : \mathcal{A} \to K$. Its complexification $\mathcal{W}_{\mathbb{C}} : S \to K_{\mathbb{C}}$ is well-defined, and one may prove [19]:

Theorem 1 *The map $\mathcal{W}_{\mathbb{C}}$, restricted to $J^{-1}(0)$ is $\mathcal{G}$-invariant, and quotients to a symplectomorphism from $J^{-1}(0)/\mathcal{G}$ to $K_{\mathbb{C}}$.*

The reduction to T^*K was first found in [23], and proved rigorously in [24].

3. QUANTIZED MARSDEN–WEINSTEIN REDUCTION

We now wish to quantize the classical setup. Traditionally, physicists have used Dirac's method of constrained quantization, in which firstly the unconstrained phase space S is quantized into a Hilbert space S, secondly the canonical $\mathcal{G}$-action on S is quantized into a unitary representation $U(\mathcal{G})$ on $\mathcal{H}$, and thirdly, the two-step classical reduction procedure

$$S \to J^{-1}(0) \to J^{-1}(0)/\mathcal{G} \tag{3.1}$$

is replaced by the single step of forming the physical subspace $\mathcal{H}_D$ of vectors in $\mathcal{H}$ that are invariant under all $U(g)$, $g \in \mathcal{G}$. The physical inner product on $\mathcal{H}_D$ is the one inherited from $\mathcal{H}$. When $\mathcal{G}$ is connected, this is equivalent

to saying that $\mathcal{H}_D$ is the subspace of $\mathcal{H}$ that is annihilated by all quantized constraints $dU(X)$, $X \in \mathfrak{g}$. Hence Dirac quantized the first step of the classical reduction procedure.

This method leads to severe mathematical difficulties, mostly because $\mathcal{H}_D$ may well be empty, even when S^0 is not. While Dirac was right in quantizing only half of the classical constraint algorithm, in our opinion he misidentified which half. It turns out that a mathematically correct theory is possible if one quantizes the second step of quotienting by the $\mathcal{G}$-action. Namely, as explained in detail in our talk at the 1995 Białowieża Workshop [18] (also cf. [19]), the mathematical analogy between symplectic reduction and induced representations of C^*-algebras eventually leads to the following algorithm of constrained quantization.

Firstly, the classical constraints define a positive definite quadratic form $(\, , \,)_0$ that is defined on a dense subspace $\mathcal{D} \subset \mathcal{H}$ (the passage from $\mathcal{H}$ to $\mathcal{D}$ is a purely technical functional-analytic matter, which should not be compared with the passage from S to the constraint manifold $J^{-1}(0)$ in the classical theory). Secondly, writing $\mathcal{N}$ for the null space of the form $(\, , \,)_0$, the physical Hilbert space $\mathcal{H}^0$ is the completion of $\mathcal{D}/\mathcal{N}$ in the inner product inherited from $(\, , \,)_0$ (and not from the original inner product $(\, , \,)$), which is positive definite on $\mathcal{D}/\mathcal{N}$ precisely because its null vectors have been thrown out. A weak observable is an operator A on $\mathcal{H}$ that leaves $\mathcal{D}$ stable and satisfies

$$(\Psi, A\Phi)_0 = (A^*\Psi, \Phi)_0. \tag{3.2}$$

This property implies that $A\mathcal{N} \subseteq \mathcal{N}$, so that A defines an operator A^0 on the quotient $\mathcal{D}/\mathcal{N}$. This operator (which may be extended to all of $\mathcal{H}^0$ under a suitable boundedness assumption) is the physical observable defined by A. More precisely, when $V : \mathcal{D} \to \mathcal{D}/\mathcal{N} \subseteq \mathcal{H}^0$ is the canonical projection, one has

$$VA\Psi = A^0 V\Psi \tag{3.3}$$

for all $\Psi \in \mathcal{D}$. We call V the quantum reduction map. By definition of the inner product $(\, , \,)^0$ in $\mathcal{H}^0$, it satisfies

$$(V\Psi, V\Phi)^0 = (\Psi, \Phi)_0. \tag{3.4}$$

It follows that one need not work with the abstract definition of $\mathcal{H}^0$; in particular, it is not necessary to compute the null space $\mathcal{N}$. Given any Hilbert space $\tilde{\mathcal{H}}^0$ and a linear map $V : \mathcal{D} \to \tilde{\mathcal{H}}^0$ which satisfies (3.4) and has dense range, one may define A^0 as an operator on $\tilde{\mathcal{H}}^0$ by means of (3.3), defining the physical theory on $\tilde{\mathcal{H}}^0$ rather than on $\mathcal{H}^0$. The corresponding representation $A \mapsto A^0$ of the algebra of all weak observables is equivalent to the one originally defined on $\mathcal{H}^0$.

In general, the quadratic form $(\, , \,)_0$ is not even closable. In the rare case that it is bounded, so that one may put $\mathcal{D} = \mathcal{H}$, one obtains $\mathcal{N}$ as a closed

subspace of $\mathcal{H}$, and $\mathcal{H}^0$ is isomorphic to $\mathcal{H}_D = \mathcal{N}^\perp$. This occurs when $\mathcal{G}$ is compact, in which case one finds [18, 19]

$$(\Psi, \Phi)_0 = \int_{\mathcal{G}} dg\, (\Psi, U(g)\Phi), \tag{3.5}$$

where dg is the normalized Haar measure on $\mathcal{G}$. One may bring the integration inside the inner product, and using the expression $p_{\mathrm{id}} = \int_{\mathcal{G}} dg\, U(g)$ for the projection on the trivial subrepresentation of a compact group yields

$$(\Psi, \Phi)_0 = (p_{\mathrm{id}}\Psi, p_{\mathrm{id}}\Phi). \tag{3.6}$$

Hence $\mathcal{N} = (p_{\mathrm{id}}\mathcal{H})^\perp$, so that

$$\mathcal{H}^0 = p_{\mathrm{id}}\mathcal{H} = \mathcal{H}_D. \tag{3.7}$$

This case is physically relevant to gauge theories on a finite lattice, where the $\mathcal{G}$-integration has indeed been standard practice from the start.

However, when $\mathcal{G}$ is noncompact but still locally compact, one generically finds that the form (3.5) is only defined on a proper dense subspace of $\mathcal{H}$, so that the general procedure just described has to be followed. This is a typical situation where Dirac's method breaks down but the improved method still works. Let us note that for each $\chi \in \mathcal{G}$ the operator $U(\chi)$ is a weak observable, since (3.2) holds; in fact, assuming that the Haar measure is right-invariant, both sides of (3.2) are equal to

$$(\Psi, U(\chi)\Phi)_0 = (\Psi, \Phi)_0. \tag{3.8}$$

This implies that the physical observable defined by $U(\chi)$ is simply $U(\chi)^0 = \mathbb{I}$ (the unit operator on the physical state space $\mathcal{H}^0$). This property guarantees the gauge invarince of the physical theory.

A gauge group (in the continuum) is not even locally compact, so in the absence of a Haar measure it is not clear what (3.5) should mean. Before turning to this problem, let us write down the relevant data for Yang–Mills theory on the circle [20, 19]. Since the classical phase space (2.2) is a Hilbert space, its quantization is the bosonic Fock space $\mathcal{H} = \exp(S)$. This space contains "exponential vectors" $|Z\rangle$ that are parametrized by $Z \in S$ and defined by [19]

$$|Z\rangle = \sum_{l=0}^{\infty} \frac{\otimes^l Z}{\sqrt{l!}} = \Omega + Z + \frac{Z \otimes Z}{\sqrt{2!}} + \cdots. \tag{3.9}$$

The square-roots are explained by the property

$$(\langle W|, |Z\rangle)_{\exp(S)} = \exp(W, Z)_S. \tag{3.10}$$

In the case at hand, the general class of representations of gauge groups considered in [10, 2] may be written as (henceforth omitting the suffix S)

$$U(g)|Z\rangle = e^{-\frac{1}{2}\|\dot{g}\|^2+(\dot{g},Z)}|Z^g\rangle. \tag{3.11}$$

In other words, the vector $|Z\rangle$ transforms classically, up to a prefactor that guarantees unitarity (see [14] for a non-unitary approach to the reduction problem). Various arguments indicate that $U(\mathcal{G})$ is indeed the correct quantization of the classical $\mathcal{G}$-action on S [7, 20, 19].

4. HILBERT SUBSPACES AND WIENER MEASURE

We are now going to make sense of the expression (3.5) for our gauge group $\mathcal{G}$ of Yang–Mills theory on a circle (defined below (2.2)). It turns out that, instead of using the non-existent invariant Haar measure on $\mathcal{G}$, we can define the form $(\ ,\ )_0$ in terms of a Gaussian measure μ on a certain completion $\mathcal{G}^-$ of $\mathcal{G}$. The special nature of the gauge group $\mathcal{G}$ as a subgroup of $\mathcal{G}^-$ lies in the fact that μ is quasi-invariant under translations by $\chi \in \mathcal{G}^-$ iff χ lies in $\mathcal{G}$. This property will guarantee that (3.8) holds, implying gauge invariance of the physical theory.

It is enlightening to present the general mathematical context of this subtle phenomenon. The following concept was introduced by Laurent Schwartz [26, 27]. A Hilbert subspace of a topological vector space $\mathcal{V}$ is a Hilbert space $\mathcal{H}$ with continuous linear injection $\mathcal{H} \hookrightarrow \mathcal{V}$. In other words, $\mathcal{H}$ is a continuously embedded subspace of $\mathcal{V}$. The Riesz–Fischer theorem then leads to an antilinear map $\theta \mapsto \tilde{\theta}$ from $\mathcal{V}^*$ to $\mathcal{H}$ (and hence to $\mathcal{V}$), defined by the property $\theta(w) = (\tilde{\theta}, w)$ for all $w \in \mathcal{H}$. One obtains a positive sesquilinear form Q on $\mathcal{V}^*$ by

$$Q(\theta, \eta) = (\tilde{\eta}, \tilde{\theta}). \tag{4.1}$$

For a simple example, consider the case that $\mathcal{V} = \mathcal{V}^* = \mathcal{H} \oplus \mathcal{K}$ is itself a Hilbert space, with the obvious embedding of $\mathcal{H}$. Then $\tilde{\theta} = p_{\mathcal{H}}\theta$ and $Q(\theta, \eta) = (\theta, p_{\mathcal{H}}\eta)$, where $p_{\mathcal{H}}$ is the orthogonal projection onto $\mathcal{H}$.

Now suppose that $\mathcal{V}$ carries a Radon measure μ whose Fourier transform is given by

$$\int_{\mathcal{V}} d\mu(v)\, e^{i\theta(v)} = e^{-\frac{1}{2}Q(\theta,\theta)}, \tag{4.2}$$

where $\theta \in \mathcal{V}^*$. A measure with this property is called Gaussian, with covariance Q, and is uniquely determined by its Fourier transform (4.2).

The general Cameron–Martin theorem [30] describes the behaviour of μ under translation. Recall that two measures are equivalent if they have the same null sets, and disjoint if their supports are disjoint.

Theorem 2 *Let $\mathcal{H}$ be a real Hilbert subspace of a quasi-complete locally convex Hausdorff vector space $\mathcal{V}$.*

1. *The map $\tilde{\theta} \mapsto \hat{\theta}$ from $\tilde{\mathcal{V}}^*$ to $L^2(\mathcal{V}, \mu)$, defined by $\hat{\theta}(v) = \theta(v)$, is well-defined and isometric, so that it extends to an isometry $w \mapsto \hat{w}$ from $\mathcal{H}$ to $L^2(\mathcal{V}, \mu)$. For each $w \in \mathcal{H}$ this defines $\hat{w}$ as an element of $L^2(\mathcal{V}, \mu)$; we write $(w, v) = \hat{w}(v)$, which makes sense for almost all $v \in \mathcal{V}$ with respect to μ.*

2. *The translate of μ by $w \in \mathcal{V}$ is disjoint from μ when $w \notin \mathcal{H}$, and equivalent to μ when $w \in \mathcal{H}$, with Radon–Nikodym derivative*

$$d\mu(v + w) = e^{-\frac{1}{2}(w,w)-(w,v)} d\mu(v). \tag{4.3}$$

In the example $\mathcal{V} = \mathcal{H} \oplus \mathcal{K}$ considered above, assume that $\mathcal{H} \simeq \mathbb{R}^n$ is finite-dimensional (for otherwise μ will not exist). Then μ is simply the standard Gaussian measure on $\mathbb{R}^n$ supported on the hyperplane $\mathcal{H}$ in $\mathcal{V}$, and the claims of the theorem are obvious: a translation by $w \in \mathcal{H}$ doesn't change this hyperplane, whereas translating by $w \notin \mathcal{H}$ moves it to a hyperplane that is disjoint from $\mathcal{H}$.

Let us note that L.P. Gross's approach to measures on infinite-dimensional vector spaces in terms of "measurable norms" (see [16]) may be seen in the above light, taking $\mathcal{V}$ to be a Banach space. The following example is a case in point (cf. [30]).

Take

$$\mathcal{H} = L^2([0, 1], \mathbb{R}^n) \tag{4.4}$$

and

$$\mathcal{V} = C([0, 1], \mathbb{R}^n)_0, \tag{4.5}$$

seen as a Banach space in the supremum-norm; the suffix $_0$ indicates that elements of $\mathcal{V}$ are (interpreted as) paths A in $\mathbb{R}^n$ that start at 0 at $t = 0$ (and continue until $t = 1$). The anti-derivative $\mathcal{P}A(t) = \int_0^t ds\, A(s)$ embeds $\mathcal{H}$ continuously into $\mathcal{V}$. The quadratic form Q on the dual $\mathcal{V}^*$, consisting of the signed Radon measures on $[0, 1]$ tensored with $\mathbb{R}^n$, reads

$$Q(\theta, \eta) = \sum_{i=1}^{n} \int_0^1 \int_0^1 d\theta_i(s)\, d\eta_i(t) \min(s, t). \tag{4.6}$$

The Gaussian measure characterized by (4.2) indeed exists, being nothing but the Wiener measure μ_W. Theorem 2 then reduces to the original Cameron–Martin theorem [8]. The Hilbert subspace $\mathcal{P}L^2([0, 1], \mathbb{R}^n)$ of paths with finite energy is known as the Cameron–Martin subspace of $C([0, 1], \mathbb{R}^n)_0$, which plays a central role in infinite-dimensional stochastic analysis [21].

Physicists sometimes write the Wiener measure as

$$d\mu_W[x(\cdot)] = N \left(\prod_{t=0}^{1} dx(t) \right) e^{-\frac{1}{2}\int_0^1 \dot{x}^2}, \tag{4.7}$$

where neither the infinite normalization constant N nor the infinite product makes mathematical sense. Moreover, the L^2-norm of $\dot{x}$ is finite iff x lies in the Cameron–Martin subspace, which is unfortunately of μ_W-measure zero. Nonetheless, assuming that the "Lebesgue measure" $dx = \prod_t dx(t)$ is translation-invariant, one may verify (4.3) from (4.7). We will make heuristic use of (4.7) also in the next section.

5. GAUGE INVARIANCE FROM THE CAMERON–MARTIN THEOREM

In order to apply the results of the preceding section to Yang–Mills theory, we should deal with the fact that our gauge group $\mathcal{G} = \mathcal{H}_1(\mathbb{T}, K)$ is not a linear space. Nonetheless, it will play the role of the Cameron–Martin subspace of $\mathcal{G}^- = LK = C(\mathbb{T}, K)_e$, the group of all based continuous loops in K, which is the analogue of $\mathcal{V}$. This time the inclusion $\mathcal{G} \hookrightarrow LK$ continuously injects a Hilbert manifold into a Banach manifold (the pertinent tangent spaces are $T_e\mathcal{G} = \mathfrak{g} = \mathcal{H}_1(\mathbb{T}, \mathfrak{k})$, the Sobolev space of continuous loops in $\mathfrak{k}$ with derivative in L^2, and $T_e(LK) = C(\mathbb{T}, \mathfrak{k})_0$, the Banach space of continuous loops in $\mathfrak{k}$ with supremum-norm, both classes of loops starting at 0).

The passage from the space $C([0,1], \mathfrak{k})_0$ of the preceding section (in which we put $\mathfrak{k} \simeq \mathbb{R}^n$) to the based loop group LK is accomplished in two steps. Firstly, Ito's map $\hat{\mathcal{I}} : C([0,1], \mathfrak{k})_0 \to C([0,1], K)_e$ is defined by [9, 22]

$$\hat{\mathcal{I}}_X(t) = \lim_{N\to\infty} \prod_{n=0}^{N-1} \operatorname{Exp}\left[X\left((1-\frac{n+1}{N})t\right) - X\left((1-\frac{n}{N})t\right)\right]. \tag{5.1}$$

It can be shown that the limit exists for almost every X with respect to the Wiener measure μ_W. The image of μ_W under Ito's map is the Wiener measure μ_W^{CK} on $C([0,1], K)_e$. Ito's map is a bijection up to null sets of μ_W and μ_W^{CK}. Restricted to $\mathcal{P}L^2([0,1], \mathfrak{k}) \subset C([0,1], \mathfrak{k})_0$, we may write Ito's map in terms of an incomplete Wilson loop as $\hat{\mathcal{I}} \circ \mathcal{P} = \hat{\mathcal{W}}$, where $\hat{\mathcal{W}} : L^2([0,1], \mathfrak{k}) \to C([0,1], K)_e$ is defined by

$$\hat{\mathcal{W}}(A) : t \mapsto P \operatorname{Exp}\left(-\int_0^t ds\, A(s)\right); \tag{5.2}$$

cf. (2.4). This may seem pointless, because $\mathcal{P}L^2([0,1], \mathfrak{k})$ has zero Wiener measure, but the point is that, using the technique of stochastic differential equations [9, 22, 14], eq. (5.2) may be extended from the domain $L^2([0,1], \mathfrak{k})$ to the domain of generalized derivatives of functions in $C([0,1], \mathfrak{k})$.

Secondly, the measure μ_W^{CK} on $C([0,1], K)_e$ is conditioned so as to become supported on LK, yielding the Wiener measure μ_W^{LK} on LK. The analogue of Theorem 2.2 [9, 22, 25] then reads

Theorem 3 *The translate of μ_W^{LK} by $\chi \in LK$ is disjoint from μ_W^{LK} when $\chi \notin \mathcal{G}$, and equivalent to μ when $\chi \in \mathcal{G}$, with Radon–Nikodym derivative*

$$d\mu_W^{LK}(g\chi) = e^{-\frac{1}{2}\|\dot{\chi}\|^2 - (\dot{g}, \mathrm{Ad}(\chi)\dot{\chi})} d\mu_W^{LK}(g), \tag{5.3}$$

where the second term in the exponential is defined as in Theorem 2.1.

We now return to the problem of making sense of the expression (3.5) for Yang–Mills theory, or, more precisely, of defining a quadratic form $(\,,\,)_0$ on some dense domain $\mathcal{D} \subset \exp(\mathcal{A})$ that satisfies (3.8). They key heuristic fact is that (3.11) leads to the expression

$$(\langle W|, U(g)|Z\rangle) = e^{-\frac{1}{2}\|\dot{g}\|^2} e^{(W, Z^g) + (\dot{g}, Z)}. \tag{5.4}$$

Combined with (4.7) and (3.5), this expression motivates the definition of $(\,,\,)_0$ on the exponential vectors (3.9) by

$$(\langle W|, |Z\rangle)_0 = \int_{LK} d\mu_W^{LK}(g)\, e^{(W, Z^g) + (\dot{g}, Z)}. \tag{5.5}$$

Here the expressions of the type $(\dot{g}, Z)$, which in (5.4) were well-defined for $g \in \mathcal{H}_1(S^1, K)$ as inner products, make sense for general $g \in LK$ by Theorem 2.1. The domain $\mathcal{E}$ of finite linear combinations of the coherent states (3.9) is dense in $\exp(\mathcal{A})$; it may be shown that (5.5) may be consistently extended to $\mathcal{E}$ (this is nontrivial, because the coherent states are overcomplete). Thus we define $(\,,\,)_0$ on $\mathcal{D} = \mathcal{E}$ by sesquilinear extension of (5.5).

The rigorous justification of (5.5), which in itself has been obtained by a heuristic argument, is that (3.8) is satisfied for all $\chi \in \mathcal{G}$; this is an easy consequence of Theorem 3. As explained in section 3, this implies that the physical theory is gauge invariant under the gauge group $\mathcal{G}$. The gauge group of the theory is $\mathcal{G}$ rather than the auxiliary device LK: the representation U given in (3.11) cannot even be extended from $\mathcal{G}$ to LK, and even if it could, Theorem 3 would make it clear that the fundamental property (3.8) only holds for $\mathcal{G}$ rather than for all of LK.

6. HALL'S COHERENT STATES

Now that we have found the quadratic form $(\,,\,)_0$ for Yang–Mills theory on a circle, we may try to compute the physical Hilbert space $\mathcal{H}^0$ and the associated representation of the weak physical observables; see section 3. Here Hall's coherent states turn out to play a crucial role. Let us first, however, review the general notion of a coherent state [15, 19].

Definition 1 *Given a manifold S, a subset $I \subset \mathbb{R}\backslash\{0\}$ having 0 as an accumulation point, and a family $\{\mathcal{H}_\hbar\}_{\hbar \in I}$ of Hilbert spaces, a system of coherent states is a collection $\{\Psi_\hbar^\sigma \in \mathcal{H}_\hbar\}_{\hbar \in I}^{\sigma \in S}$ of unit vectors, along with a set $\{\mu_\hbar\}_{\hbar \in I}$ of Radon measures on S, such that*

1. *for each $\hbar \in I$ and all $\Psi \in \mathcal{H}_\hbar$ one has the completeness property*
$$\int_S d\mu_\hbar(\sigma)\, |(\Psi^\sigma_\hbar, \Psi)|^2 = 1; \tag{6.1}$$
2. *for each $\hbar \in I$ the map from S to the projective space $\mathbb{P}\mathcal{H}_\hbar$ defined by projecting $\sigma \mapsto \Psi^\sigma_\hbar$ is continuous;*
3. *for fixed ρ and σ the function $\hbar \mapsto |(\Psi^\sigma_\hbar, \Psi^\rho_\hbar)|^2$ is continuous, with classical limit*
$$\lim_{\hbar \to 0} |(\Psi^\sigma_\hbar, \Psi^\rho_\hbar)|^2 = \delta_{\rho\sigma}. \tag{6.2}$$

For example, one may take S to be a finite-dimensional Hilbert space $S \simeq \mathbb{C}^n$, $I = \mathbb{R}_{>0}$, $\mathcal{H}_\hbar = \exp(S)$ (independent of $\hbar$), $\mu_\hbar$ equal to $(2\pi\hbar)^{-n}$ times Lebesgue measure, and
$$_{\text{Fock}}\Psi^\sigma_\hbar = e^{-\frac{1}{2}(\sigma,\sigma)/\hbar} |\sigma/\sqrt{\hbar}\rangle, \tag{6.3}$$
cf. (3.9). It is follows from (3.10) that
$$|(_{\text{Fock}}\Psi^\sigma_\hbar, {}_{\text{Fock}}\Psi^\rho_\hbar)|^2 = e^{-|\rho-\sigma|^2/\hbar}, \tag{6.4}$$
which implies all properties in Definition 1. In case that S is an infinite-dimensional Hilbert space, this discussion is still valid if all reference to the measures $\mu_\hbar$ (and therefore condition (6.1)) is omitted. Hence the exponential vectors (3.10) are essentially coherent states with $\hbar = 1$, up to normalization; the normalized coherent states are given by (6.3).

Hall's coherent states were introduced in [11], but their interpretation as coherent states satisfying Definition 1 only became clear from [12, 13]. Their definition involves the fundamental solution $\rho(x,t)$ of the heat equation $df/dt - \frac{1}{2}\Delta_K f = 0$ on a compact connected Lie group K, as well as the fundamental solution $\rho_{\mathbb{C}}(\sigma, t)$ of the heat equation $df/dt - \frac{1}{2}\Delta_{K_{\mathbb{C}}} f = 0$ on $K_{\mathbb{C}}$; here Δ_K and $\Delta_{K_{\mathbb{C}}}$ are the Laplacians on K and on $K_{\mathbb{C}}$, respectively; cf. [11]. Hall proved in [11] that ρ may be analytically continued from K to $K_{\mathbb{C}}$; we call this continuation $\rho^{\mathbb{C}}$.

Definition 2 *Let K be a compact connected Lie group. In the context of Definition 1 one takes $S = K_{\mathbb{C}}$ (cf. section 2), $I = \mathbb{R}_{>0}$, $\mathcal{H}_\hbar = L^2(K)$ (defined with respect to the normalized Haar measure, and independent of $\hbar$), $\mu_\hbar$ as defined through its Radon–Nikodym derivative with respect to the Haar measure $d\sigma$ on $K_{\mathbb{C}}$ by*
$$d\mu_\hbar(\sigma) = d\sigma \int_K dk\, \rho_{\mathbb{C}}(k^{-1}\sigma, \hbar), \tag{6.5}$$
and finally
$$_{\text{Hall}}\Psi^\sigma_\hbar : k \mapsto N_\hbar \rho^{\mathbb{C}}(k^{-1}\sigma, \hbar), \tag{6.6}$$
where $N_\hbar$ is a certain (σ-independent) normalization constant guaranteeing that $_{\text{Hall}}\Psi^\sigma_\hbar$ is a unit vector in $L^2(K)$.

It is possible to use this definition for $K = \mathbb{R}^n$, so that $K_{\mathbb{C}} = \mathbb{C}^n$; using the explicit expressions for the heat kernels on $\mathbb{R}^n$ and $\mathbb{C}^n$, one recovers the coherent states (6.3) in the Bargmann–Fock representation. See [12, 13].

7. PUNCH LINE

All threads now come together.

Theorem 4 *In the discussion following (3.4) we take $(\,,\,)_0$ and $\mathcal{D}$ as defined in (5.5) and subsequent text, and $\tilde{\mathcal{H}}^0 = L^2(K)$.*

- *The quantum reduction map $V : \mathcal{D} \to L^2(K)$, given by linear extension of*
$$V\,{}_{\text{Fock}}\Psi_1^Z = {}_{\text{Hall}}\Psi_{\frac{1}{2}}^{\mathcal{W}_{\mathbb{C}}(Z)}, \tag{7.1}$$
satisfies (3.4). Here $Z \in S = L^2(\mathbb{T}, \mathfrak{k}_{\mathbb{C}})$ (cf. (2.2)), and the complexified Wilson loop $\mathcal{W}_{\mathbb{C}}(Z) \in K_{\mathbb{C}}$ is defined below (2.4).

- *For each $\chi \in \mathcal{G}$ one has*
$$VU(\chi)\,{}_{\text{Fock}}\Psi_1^Z = {}_{\text{Hall}}\Psi_{\frac{1}{2}}^{\mathcal{W}_{\mathbb{C}}(Z)}. \tag{7.2}$$

- *Let $f \in C^\infty(K)$, defining the function $\mathcal{W}_f : A \mapsto f(\mathcal{W}(A))$ on S. These functions can be quantized by certain operators $\mathcal{Q}(\mathcal{W}_f)$ on $\exp(S)$; see [19]. One then has*
$$V\mathcal{Q}(\mathcal{W}_f)\,{}_{\text{Fock}}\Psi_1^Z = f\,{}_{\text{Hall}}\Psi_{\frac{1}{2}}^{\mathcal{W}_{\mathbb{C}}(Z)}. \tag{7.3}$$

Hence the physical Hilbert space of Yang–Mills theory on the circle may be identified with $L^2(K)$, on which the gauge group acts trivially, and functions of the Wilson loop act as multiplication operators.

The calculations leading to (7.1) and (7.1) are presented in [33, 34, 19]. The proof that Hall's coherent states are total in $L^2(K)$ is in [11]; this is necessary in view of the discussion after (3.4). Finally, (7.2) is a direct consequence of (3.8); see the end of the previous section.

There is a statement similar to (7.3) in which the physical Hamiltonian $-\frac{1}{2}\Delta_K$, is derived from the Hamiltonian of the unconstrained theory; see [34, 35]. However, the latter is an extremely involved operator, and it seems that the techniques developed in [14] are more suitable to deal with it than ours.

It may be argued that we have quantized a rather simple model in an incredibly complicated way. However, in higher-dimensional gauge theories the reduced phase space is not known explicitly, and one must deal with the constraints in some way. We hope that both our general method of constrained

quantization and the special techniques used to apply this method to Yang–Mills theory on a circle can be generalized to higher dimensions. The essential task will be to realize the gauge group as a generalized Cameron–Martin subspace of some enlargement of it, along with a suitable generalization of Theorem 3 that eventually guarantees the gauge invariance of the physical quantum theory. It seems to us that the probabilistic literature offers some hope for this to be possible.

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